# Introduction to Computational Complexity

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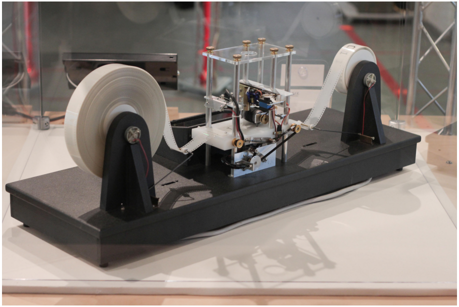
# Computability

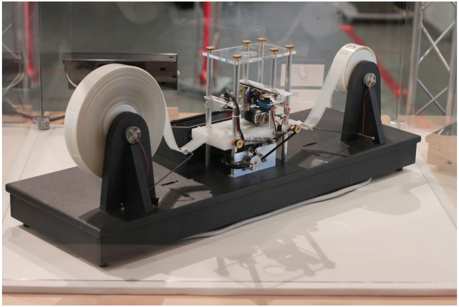
Central questions of theoretical computer science are connected to topics such as which problems are **computable** and **how efficiently**.



- Example: 5 states, 5 symbols, single tape

# Computability

Central questions of theoretical computer science are connected to topics such as which problems are **computable** and **how efficiently**.



- Example: 5 states, 5 symbols, single tape
- A fascinating thing about computational complexity is that we basically all agree on what can be computed.
- How efficiently? Much less clear.

# What does it mean to compute

By computing, we mean to evaluate a specific function
$f : \{0,1\}^* \mapsto \{0,1\}^*$ ($S^* = \bigcup_{n\geq 0} S^n$ is the set of all finite strings over $S$).
e.g., $f(0110...011) = 1101100...10101$, also note that $f : \mathbb{N}_0 \mapsto \mathbb{N}_0$

This can be done by a procedure called **an algorithm**:

## Algorithm (informal)

An algorithm $A$ given by a finite description computes a function $f$ if:

1. $A$ provides the output of $f$ by following a **finite procedure** described by unambiguous elementary steps.
2. $A$ performs a **finite number of steps** (with no particular bound on the storage space used).

- Note that if $A$ is representable by a finite bit string, then also $A$ corresponds to some $a \in \mathbb{N}_0$.
- An algorithm is essentially what we understand as a **computable function**.
- Notice that we do not speak about how efficient the computation of the function should be.

# Power of different computational models

Power of a computational model:

- The sequence of steps that define the algorithm is executed by a **computational model** (e.g., mechanical machine, computer).
- But the computational model that manipulates with symbols should have a certain complexity (i.e., sufficiently powerful instruction set) to calculate (at least some) computable functions, right?
  - What operations are allowed? Random-access memory? Stack only? Conditional branching?
- What makes a computer a "universal" one?

# Turing machine is the universal model of computation

- **Turing machine (TM)**: the universal model of computation.
  - Abstract machine described by Alan Turing that reads symbols, changes its state, **rewrites symbols** on the tape, **moves the tape**.
  - Our computers $\approx$ implementation of TM.

**Church-Turing conjecture** (1936)

All computable functions are exactly Turing-computable (although not necessarily very efficiently).

- Not so obvious: we have examples that are known to be strictly less powerful: finite state machines, push-down automata, ...
- On the other hand, different computational models (e.g., $\lambda$-calculus, TM with access to random bits) do not seem to be more powerful.
- We equate the (intuitive) concept of computable function with Turing-computable, which can be precisely defined.
- We can restrict our study of computational problems under the TM.

# Computational problems

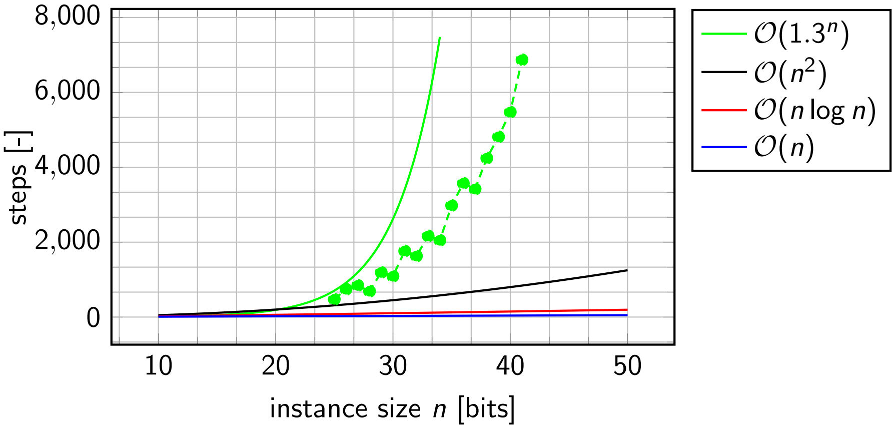
- Computational problems can be seen as relations between the inputs (instances) and outputs (solutions).
- $x$ encoding of the instance, $y$ encoding of the solution over some alphabet, $S = \{0, 1\}$, $S^* = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, \ldots\}$.
- Let $R(x, y) \subseteq S^* \times S^*$ be a relation. Each $R$ defines a computational problem:

## Types of computational problems

- **Decision problems**
  - Given $x$, determine if there is $y$ satisfying $R(x, y)$?
- **Search problems**
  - Given $x$, find $y$ such that $R(x, y)$ or state it does not exist.
- **Optimization problems**
  - Given $x$, find $y$ such that $R(x, y)$ minimizing function $c(x, y)$ or say no such $y$ exists.
- **Function problems**
  - Compute value of $f(x)$.

# Complexity of problems

- Not all $R$ (computational problems) are equally difficult.
- We can measure the difficulty of the problem by the number of steps $T(n)$ the best-known algorithm $A$ on a TM needs to solve the problem $R$ for a given input length $n = |x|$ in **the worst-case**.



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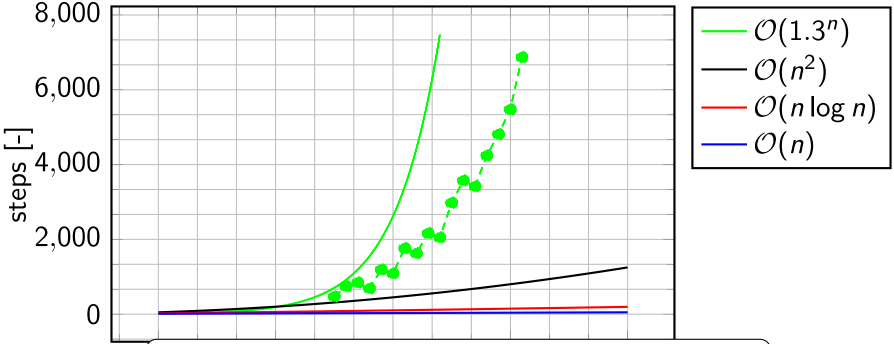


**Computers are not to rescue:** electron-sized transistor, clock time ≈ time of light between atoms, Earth-sized computer needs billions of years to brute force 300-bit solution.

# Problems, instances and algorithms: summary

- A computational **problem** is a relation over **instances** and **solutions**.
- To solve problems, we develop **algorithms** with certain **time complexity**.
  - Measured in terms of the worst-case number of steps $T(n)$ over all instances of length $n$.
- The existence of an algorithm with given time complexity $\mathcal{O}(T(n))$ is a witness of the **problem** being in certain **complexity class**.
  - People started to categorize problems into a taxonomy.
- It turned out that there is a fundamental barrier between the polynomially solvable problems and the others.
  - In practice, we usually get low-degree polynomial algorithms or exponential ones (or even worse).

It motivates us to study which problems fall into the "good" category and which fall into "naughty".

# Efficiently solvable: P

- We will use decision problems (yes/no) to demonstrate the most prominent complexity classes.
  - Similar can also be done for optimization problems, with a few definition adjustments.

## Definition: Class P

The set of problems that are **solvable** in a polynomial time.

- For every $x \in \{0,1\}^*$ they state if $\exists y \in \{0,1\}^* : R(x,y)$ or no.
- Admit $\mathcal{O}(\text{poly}(n))$, $n = |x|$ algorithm, e.g. $\mathcal{O}(n^2)$, $\mathcal{O}(n \log n)$.

## Examples

- Integer number problems: Addition, Multiplication, Primality test,...
- Graph problems: Topological Sorting, Minimum Spanning Tree, ...
- Miscellaneous problems: Discrete Fourier Transform, Linear Programming, ...

# Efficiently checkable: NP

**Definition: Class NP (non-deterministic polynomial)**

The set of decision problems whose solutions are **checkable** in a polynomial time.

- What do we mean by checkable?

# Efficiently checkable: NP

## Definition: Class NP (non-deterministic polynomial)

The set of decision problems whose solutions are **checkable** in a polynomial time.

- What do we mean by checkable?
- **YES**-instances have so-called **polynomial certificates** (or witnesses, proofs): e.g., $|y| \leq \text{poly}(|x|)$.
- Given poly-sized certificate $y$, one can in $\text{poly}(|x| + |y|)$ time verify that indeed $(x, y) \in R$.
- We do not know whether they admit a $\mathcal{O}(\text{poly}(|x|))$ algorithm, but we know that their solution is verifiable by $\mathcal{O}(\text{poly}(|x|))$ algorithm.

## Examples

- Integer number problems: Sudoku, Knapsack, 2-Partition, ...
- Graph problems: Travelling Salesman, $k$-coloring, ...
- Miscellaneous problems: Linear equations with absolute values, Control theory: constrained state-space feedback

# SubsetSum is in NP

## Definition: SubsetSum problem

- **Instance:** A (multi)-set of $n$ non-negative integers $A = \{a_1, \dots, a_n\}$ and a non-negative integer $W$.
- **Decision:** Is there a subset $S \subseteq A$ such that $\sum_{a_i \in S} a_i = W$?

## Example 1: YES-instance

$A = \{1, 1, 2, 3, 7, 9\}$, $W = 6$. The answer is **YES**: $S = \{1, 2, 3\}$.

- I claim $(A, W)$ is YES-instance. This is a poly-sized proof: $S$.

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## Example 2: NO instance

$A = \{3, 5, 5, 6, 8, 10\}$, $W = 25$. The answer is **NO**.

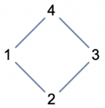
- I claim $(A, W)$ is NO-instance. I do not have a short proof.

Generally, short certificates (proofs) of **NO**-instances may not exist.

# Graph Isomorphism is in NP

## Example: Graph Isomorphism problem

- Given two graphs $G$ and $H$, decide if $G$ is the same as $H$ up to the vertex labelling.
- A $\mathcal{O}(\text{poly}(n))$ algorithm is not known.
- We have an algorithm by Babai (2015) that runs in $\mathcal{O}(\exp(\log(n)^c))$ for some constant $c > 1$, i.e., a quasipolynomial, grows smaller than an exponential.
- But its solution is checkable in a poly-time.

In[17]:= **FindGraphIsomorphism**[ 

, 

]

Out[17]= { <| 1 → 1, 2 → 3, 3 → 2, 4 → 4 |> }

- **Remark:** in contrast to NP, problems in P have polynomial certificates even for **NO**-instances.

# What is non-deterministic about NP?

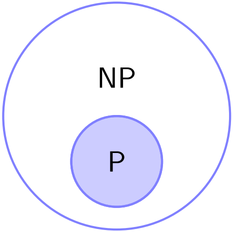
Why NP means "non-deterministic polynomial"?

- Connected with an alternative computational model, the so-called non-deterministic Turing Machine (TM).
- This abstract computational model explores all branches in your algorithm in parallel.
- This is an alternative description of class NP: the set of decision problems for which there is an algorithm that solves it in a polynomial time on a non-deterministic TM computational model.
- Useful for theoretical analysis, nobody knows how to build it physically (in contrast to the deterministic TM).
  - Quantum computers are not believed to be equivalent to non-deterministic TMs.

# P vs NP question

**How important is P vs NP question?**

- At least $1.000.000 important.
- Clay Math Institute's Millennium problems:
  - Solution smoothness of Navier–Stokes Equation
  - Poincaré Conjecture (solved)
  - Riemann Hypothesis
  - **P vs NP problem**
  - ...
- P vs NP question has wide implications to the world outside of CS: class P exactly corresponds to dynamical systems described by ODEs with polynomial RHS under a poly-length simulation (connection to control theory).



# P vs NP question

**Common belief is:**

**Conjecture**

$$P \neq NP.$$

- Likely we are not in a position to resolve this question within the next 20 years.

[1] https://www.cs.umd.edu/users/gasarch/BLOGPAPERS/pollpaper3.pdf
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- Likely we are not in a position to resolve this question within the next 20 years.
- But, we can run a survey:

| Year | 2002 | 2012 | 2019 |
|---|---|---|---|
| Thinks $P \neq NP$ | 61% | 82% | 88% |

Table: William Gasarch's survey on P vs NP[1].

- See nice explanatory video on P vs NP[2].

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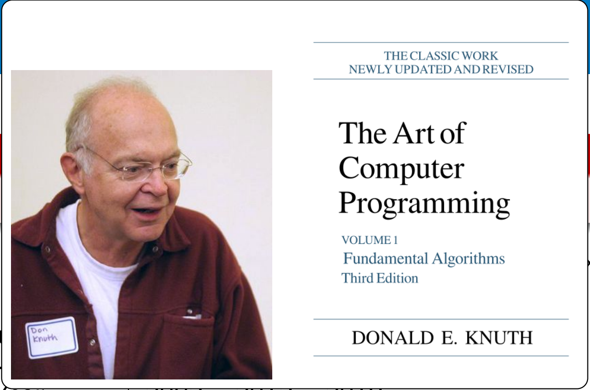


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- See nice explanatory video on P vs NP[2].
- Donald Knuth is one of the great proponents of P $=$ NP (giant alg.).

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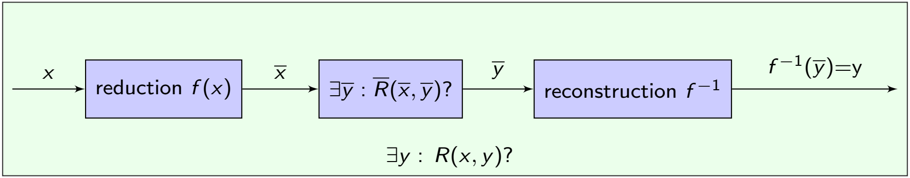
# Polynomial reductions

**Problem reductions** are one of the greatest inventions in computer science.

**Motto:** *My problems are your problems.*

- Solving a new problem $R(x, y)$ via existing problem $\overline{R}(\overline{x}, \overline{y})$:



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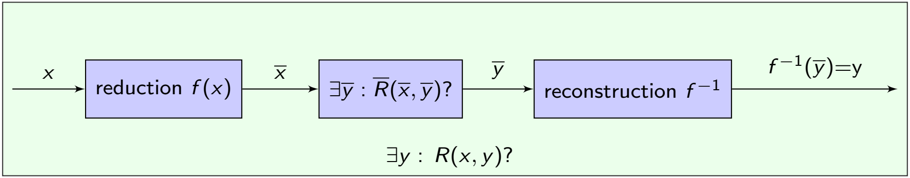


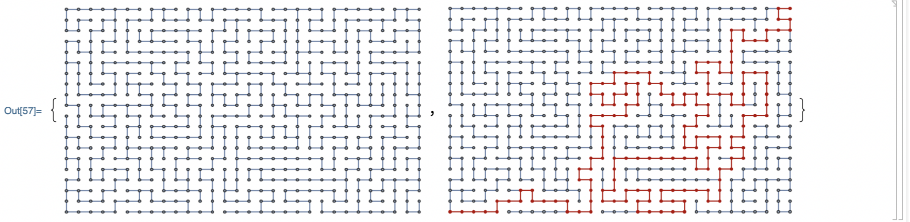

Namely, we will be interested in **polynomial-time reductions**: $R \triangleleft_P \overline{R}$.

- $f$ and $f^{-1}$ runs in a polynomial time and $\exists y : R(x, y) \iff \exists \overline{y} : \overline{R}(\overline{x}, \overline{y})$
- Preserve membership in classes P and NP: $poly(poly(n)) \in \mathcal{O}(poly(n))$.
- Useful from the practical standpoint.

# You have used polynomial reductions before

- Path with the minimum number of edges, but you only have Dijkstra.

```
{m, HighlightGraph[m, PathGraph@FindShortestPath[m, First@VertexList[m], Last@VertexList[m]]]}
```




- This is a polynomial reduction.

# Some the reductions connect different worlds

- More examples: 3CNF-SAT $\lhd_P$ SubsetSum ($R \lhd_P \overline{R}$)

## Definition: 3CNF-SAT Problem

- **Instance:** A propositional formula in conjunction normal form with clauses with 3 literals, e.g., $\phi = (x \vee \neg y \vee z) \wedge \ldots \wedge (\neg x \vee u \vee \neg v)$.
- **Decision:** Is the formula $\phi$ satisfiable?

## Example

$$\phi = (x \vee y \vee z) \wedge (x \vee y \vee \neg z) \wedge (x \vee \neg y \vee z) \wedge (x \vee \neg y \vee \neg z) \wedge \\ (\neg x \vee y \vee z) \wedge (\neg x \vee y \vee \neg z) \wedge (\neg x \vee \neg y \vee z) \wedge (\neg x \vee \neg y \vee \neg z)$$

The answer is **NO**.

# 3CNF-SAT to SubsetSum

- More examples: 3CNF-SAT $\triangleleft_P$ SubsetSum ($R \triangleleft_P \overline{R}$)

**Definition: SubsetSum problem**

- **Instance:** A (multi)-set of $n$ non-negative integers $A = \{a_1, \ldots, a_n\}$ and a non-negative integer $W$.
- **Decision:** Is there a subset $S \subseteq A$ such that $\sum_{a_i \in S} a_i = W$?

**Example**

$A = \{1, 1, 2, 3, 7, 9\}$, $W = 6$. Answer is **YES**: $S = \{1, 2, 3\}$.

- How can we use a number counting problem to solve a logic problem? These are beasts living in different realms.

# Example: 3CNF-SAT to SubsetSum

$$\phi = \underbrace{(\neg x \vee y \vee z)}_{C_1} \wedge \underbrace{(x \vee \neg y \vee z)}_{C_2} \wedge \underbrace{(\neg x \vee \neg y \vee \neg z)}_{C_3}$$

| | $x$ | $y$ | $z$ | $C_1$ | $C_2$ | $C_3$ | |
|---|---|---|---|---|---|---|---|
| $x$ | 1 | 0 | 0 | 0 | 1 | 0 | $a_1$ |
| $\neg x$ | 1 | 0 | 0 | 1 | 0 | 1 | $a_2$ |
| $y$ | 0 | 1 | 0 | 1 | 0 | 0 | $a_3$ |
| $\neg y$ | 0 | 1 | 0 | 0 | 1 | 1 | $a_4$ |
| $z$ | 0 | 0 | 1 | 1 | 1 | 0 | $a_5$ |
| $\neg z$ | 0 | 0 | 1 | 0 | 0 | 1 | $a_6$ |
| | 0 | 0 | 0 | 1 | 0 | 0 | $a_7$ |
| | 0 | 0 | 0 | 2 | 0 | 0 | $a_8$ |
| | 0 | 0 | 0 | 0 | 1 | 0 | $a_9$ |
| | 0 | 0 | 0 | 0 | 2 | 0 | $a_{10}$ |
| | 0 | 0 | 0 | 0 | 0 | 1 | $a_{11}$ |
| | 0 | 0 | 0 | 0 | 0 | 2 | $a_{12}$ |
| | 1 | 1 | 1 | 4 | 4 | 4 | $W$ |

- Notice that no carry-overs are happening.
- **Homework:** does this work for $k$CNF-SAT ($k$ literals in each clause)?

# Example: 3CNF-SAT to SubsetSum

$$\phi = \underbrace{(\neg x \vee y \vee z)}_{C_1} \wedge \underbrace{(x \vee \neg y \vee z)}_{C_2} \wedge \underbrace{(\neg x \vee \neg y \vee \neg z)}_{C_3}$$

| | $x$ | $y$ | $z$ | $C_1$ | $C_2$ | $C_3$ | |
|---|---|---|---|---|---|---|---|
| $x$ | **1** | **0** | **0** | **0** | **1** | **0** | $a_1$ |
| $\neg x$ | 1 | 0 | 0 | 1 | 0 | 1 | $a_2$ |
| $y$ | **0** | **1** | **0** | **1** | **0** | **0** | $a_3$ |
| $\neg y$ | 0 | 1 | 0 | 0 | 1 | 1 | $a_4$ |
| $z$ | 0 | 0 | 1 | 1 | 1 | 0 | $a_5$ |
| $\neg z$ | **0** | **0** | **1** | **0** | **0** | **1** | $a_6$ |
| | **0** | **0** | **0** | **1** | **0** | **0** | $a_7$ |
| | **0** | **0** | **0** | **2** | **0** | **0** | $a_8$ |
| | **0** | **0** | **0** | **0** | **1** | **0** | $a_9$ |
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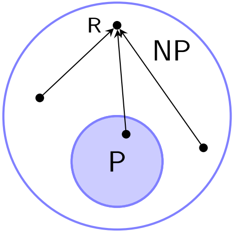
| | 1 | 1 | 1 | 4 | 4 | 4 | $W$ |

- Notice that no carry-overs are happening.
- **Homework:** does this work for $k$CNF-SAT ($k$ literals in each clause)?

# Complete problems: NP-complete

The idea of reductions can be used to identify so-called **complete problems** for the class.




## Definition: NP-complete class

Problem $R$ is NP-complete if $R \in \text{NP}$ (i.e., efficiently checkable) and for every problem $A$ :

$\forall A \in \text{NP} : A \triangleleft_P R$ (i.e., acts as a solver for class NP).

- The meaning of an NP-complete problem is that it represents a "universal" problem for NP class (can be used to solve all problems in NP).
- The first NP-complete problem was discovered by Cook (1971):
    - Proof: non-deterministic TM $\triangleleft_P$ CNF-SAT.
    - Hence, CNF-SAT acts as a solver for class NP.
- Nowadays, we know thousands of NP-complete problems.

# Example: CNF-SAT

Problem reductions are not particularly useful if they do not run in a polynomial time.

## $k$CNF-SAT Problem

- **Instance:** A propositional formula in conjunction normal form, e.g., $\phi = (x \vee \neg y \vee z) \wedge ... \wedge (\neg x \vee u \vee \neg v)$.
- **Decision:** Is the formula $\phi$ satisfiable?

## $k$DNF-SAT Problem

- **Instance:** A propositional formula in **disjunctive normal form**, e.g., $\phi = (x \wedge \neg y \wedge z) \vee ... \vee (\neg x \wedge u \wedge \neg v)$.
- **Decision:** Is the formula $\phi$ satisfiable?

## Theorem

$k$CNF-SAT is in NP-complete (non-deterministic TM reduces to poly-sized CNF formula).
$k$DNF-SAT is in P (easy algorithm).

**Reduction idea:** We have learned in TGR and LPS courses how to convert CNFs to DNFs (disjunctive normal form), and we know that DNF-SAT is solvable in a polynomial time (how?). So lets try

$$\text{kCNF-SAT} \triangleleft_P \text{kDNF-SAT}.$$

# Example: CNF-SAT

```
In[12]:= cnf = (x₁ ∨ y₁) ∧ (x₂ ∨ y₂) ∧ (x₃ ∨ y₃) ∧ (x₄ ∨ y₄) ∧ (x₅ ∨ y₅) ∧ (x₆ ∨ y₆) ∧ (x₇ ∨ y₇);
         BooleanConvert[cnf] // TraditionalForm
```

# Example: CNF-SAT

In[12]:=
```
cnf = (x1 ∨ y1) ∧ (x2 ∨ y2) ∧ (x3 ∨ y3) ∧ (x4 ∨ y4) ∧ (x5 ∨ y5) ∧ (x6 ∨ y6) ∧ (x7 ∨ y7);
BooleanConvert[cnf] // TraditionalForm
```

Out[13]//TraditionalForm=

$(x_1 \land x_2 \land x_3 \land x_4 \land x_5 \land x_6 \land x_7) \lor (x_1 \land x_2 \land x_3 \land x_4 \land x_5 \land x_6 \land y_7) \lor (x_1 \land x_2 \land x_3 \land x_4 \land x_5 \land y_6 \land x_7) \lor (x_1 \land x_2 \land x_3 \land x_4 \land x_5 \land y_6 \land y_7) \lor$

$(x_1 \land x_2 \land x_3 \land x_4 \land y_5 \land x_6 \land x_7) \lor (x_1 \land x_2 \land x_3 \land x_4 \land y_5 \land x_6 \land y_7) \lor (x_1 \land x_2 \land x_3 \land x_4 \land y_5 \land y_6 \land x_7) \lor (x_1 \land x_2 \land x_3 \land x_4 \land y_5 \land y_6 \land y_7) \lor$

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# Example: CNF-SAT

Perhaps, if the DNF reduction would be done in a more sophisticated way:

```
In[16]:= BooleanMinimize[BooleanConvert[cnf]] // TraditionalForm
```

# Example: CNF-SAT

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In[16]:= BooleanMinimize[BooleanConvert[cnf]] // TraditionalForm
```

Out[16]//TraditionalForm=

$$(x_1 \land x_2 \land x_3 \land x_4 \land x_5 \land x_6 \land x_7) \lor (x_1 \land x_2 \land x_3 \land x_4 \land x_5 \land x_6 \land y_7) \lor (x_1 \land x_2 \land x_3 \land x_4 \land x_5 \land x_7 \land y_6) \lor (x_1 \land x_2 \land x_3 \land x_4 \land x_5 \land y_6 \land y_7) \lor$$
$$(x_1 \land x_2 \land x_3 \land x_4 \land x_6 \land x_7 \land y_5) \lor (x_1 \land x_2 \land x_3 \land x_4 \land x_6 \land y_5 \land y_7) \lor (x_1 \land x_2 \land x_3 \land x_4 \land x_7 \land y_5 \land y_6) \lor (x_1 \land x_2 \land x_3 \land x_4 \land y_5 \land y_6 \land y_7) \lor$$
$$(x_1 \land x_2 \land x_3 \land x_5 \land x_6 \land x_7 \land y_4) \lor (x_1 \land x_2 \land x_3 \land x_5 \land x_6 \land y_4 \land y_7) \lor (x_1 \land x_2 \land x_3 \land x_5 \land x_7 \land y_4 \land y_6) \lor (x_1 \land x_2 \land x_3 \land x_5 \land y_4 \land y_6 \land y_7) \lor$$
$$(x_1 \land x_2 \land x_3 \land x_6 \land x_7 \land y_4 \land y_5) \lor (x_1 \land x_2 \land x_3 \land x_6 \land y_4 \land y_5 \land y_7) \lor (x_1 \land x_2 \land x_3 \land x_7 \land y_4 \land y_5 \land y_6) \lor (x_1 \land x_2 \land x_3 \land y_4 \land y_5 \land y_6 \land y_7) \lor$$
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$$(x_3 \land x_4 \land x_5 \land x_6 \land x_7 \land y_1 \land y_2) \lor (x_3 \land x_4 \land x_5 \land x_6 \land y_1 \land y_2 \land y_7) \lor (x_3 \land x_4 \land x_5 \land x_7 \land y_1 \land y_2 \land y_6) \lor (x_3 \land x_4 \land x_5 \land y_1 \land y_2 \land y_6 \land y_7) \lor$$
$$(x_3 \land x_4 \land x_6 \land x_7 \land y_1 \land y_2 \land y_5) \lor (x_3 \land x_4 \land x_6 \land y_1 \land y_2 \land y_5 \land y_7) \lor (x_3 \land x_4 \land x_7 \land y_1 \land y_2 \land y_5 \land y_6) \lor (x_3 \land x_4 \land y_1 \land y_2 \land y_5 \land y_6 \land y_7) \lor$$
$$(x_3 \land x_5 \land x_6 \land x_7 \land y_1 \land y_2 \land y_4) \lor (x_3 \land x_5 \land x_6 \land y_1 \land y_2 \land y_4 \land y_7) \lor (x_3 \land x_5 \land x_7 \land y_1 \land y_2 \land y_4 \land y_6) \lor (x_3 \land x_5 \land y_1 \land y_2 \land y_4 \land y_6 \land y_7) \lor$$
$$(x_3 \land x_6 \land x_7 \land y_1 \land y_2 \land y_4 \land y_5) \lor (x_3 \land x_6 \land y_1 \land y_2 \land y_4 \land y_5 \land y_7) \lor (x_3 \land x_7 \land y_1 \land y_2 \land y_4 \land y_5 \land y_6) \lor (x_3 \land y_1 \land y_2 \land y_4 \land y_5 \land y_6 \land y_7) \lor$$
$$(x_4 \land x_5 \land x_6 \land x_7 \land y_1 \land y_2 \land y_3) \lor (x_4 \land x_5 \land x_6 \land y_1 \land y_2 \land y_3 \land y_7) \lor (x_4 \land x_5 \land x_7 \land y_1 \land y_2 \land y_3 \land y_6) \lor (x_4 \land x_5 \land y_1 \land y_2 \land y_3 \land y_6 \land y_7) \lor$$
$$(x_4 \land x_6 \land x_7 \land y_1 \land y_2 \land y_3 \land y_5) \lor (x_4 \land x_6 \land y_1 \land y_2 \land y_3 \land y_5 \land y_7) \lor (x_4 \land x_7 \land y_1 \land y_2 \land y_3 \land y_5 \land y_6) \lor (x_4 \land y_1 \land y_2 \land y_3 \land y_5 \land y_6 \land y_7) \lor$$
$$(x_5 \land x_6 \land x_7 \land y_1 \land y_2 \land y_3 \land y_4) \lor (x_5 \land x_6 \land y_1 \land y_2 \land y_3 \land y_4 \land y_7) \lor (x_5 \land x_7 \land y_1 \land y_2 \land y_3 \land y_4 \land y_6) \lor (x_5 \land y_1 \land y_2 \land y_3 \land y_4 \land y_6 \land y_7) \lor$$
$$(x_6 \land x_7 \land y_1 \land y_2 \land y_3 \land y_4 \land y_5) \lor (x_6 \land y_1 \land y_2 \land y_3 \land y_4 \land y_5 \land y_7) \lor (x_7 \land y_1 \land y_2 \land y_3 \land y_4 \land y_5 \land y_6) \lor (y_1 \land y_2 \land y_3 \land y_4 \land y_5 \land y_6 \land y_7)$$

# Example: CNF-SAT

**Takeaways:**

- The above example shows an exponential explosion of the resulting DNF formula.
- Unfortunately, we do not know how to convert, in general, every CNF formula to DNF in a polynomial time.
- But reductions in the opposite direction, i.e., something $\triangleleft_P$ CNF-SAT, are in fact very useful:
  - formal verification: some states are not reachable within any $k$ steps
  - proof checking: Keller's conjecture[3]
  - graph coloring, ...
- CNF-SAT is both theoretically (a universal NP problem) and practically (existence of solvers) appealing.

[3] https://www.quantamagazine.org/computer-search-settles-90-year-old-math-problem-20200819/

# NP-complete: summary

NP-complete class summary:

- The set of universal (most difficult) problems for class NP.
- All known algorithms for NP-complete problems have complexity above $\mathcal{O}(poly(n))$.
  - e.g., CNF-SAT algorithm is $\mathcal{O}(1.308^n) \approx \mathcal{O}(2^{0.387n})$
- Solving efficiently one of the thousands known NP-complete problems would mean P $=$ NP.
  - Hence, if your problem is NP-complete do not hope for a poly-time algorithm.

**NP problems suspected not being NP-complete and not in P**

- Graph Isomorphism (GI)
  - We know a subexponential algorithm, but still above a polynomial complexity.
- Integer Factoring
- Computing VC (Vapnik-Chervonenkis) dimension

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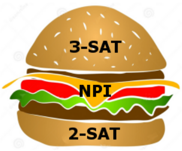


**Complexity sandwich:** But can it be filled with natural ingredients?

## NP problems suspected **not being** NP-complete and not in P

- Graph Isomorphism (GI)
  - We know a subexponential algorithm, but still above a polynomial complexity.
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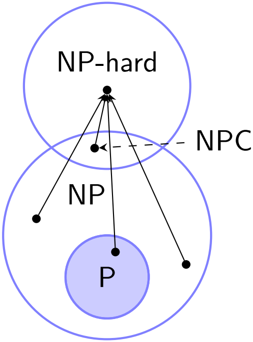
# Beyond NP-complete: NP-hard class

**Definition: NP-hard class**

Problem $R$ is NP-hard if for every problem $A$

$$\forall A \in \text{NP} : A \lhd_P R.$$

- sets a **lower bound** on the complexity of the problem (acts as a solver for class NP)
- the difference from NP-complete is that $R$ can be much harder (does not have to be in NP)




**Examples**

- every NP-complete decision problem
- optimization variants of NP-complete decision problems
- Quantified Boolean Formula satisfiability: $\forall x_1 \exists x_2 \forall x_3, \ldots : f(x_1, x_2, x_3, \ldots)$

# Conclusion

**The main takeaways:**

- Turing machine is the universal model of computation
  - Gives us a formal way studying and categorizing problems according to their complexity.
- Easy problems (P) vs. hard problems (NP-complete, NP-hard):
  - Easily solvable vs. easily checkable vs. just hard problems
  - More complexity classes live in Complexity ZOO: `https://complexityzoo.net/`.
- Polynomial reductions:
  - Using somebody else's problem to solve your problems.

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